UNIT – I
NETWORK THEOREMS

Topics: Superposition, Thevenin, Norton, Maximum Power Transfer, Millman, Tellegen, Reciprocity and Compensation theorems for D.C and sinusoidal excitations (Without Proof).

INTRODUCTION:

Electric circuit theorems are always beneficial to help find voltage and currents in multi loop circuits. These theorems use fundamental rules or formulas and basic equations of mathematics to analyze basic components of electrical or electronics parameters such as voltages, currents, resistance, and so on.

Mesh and nodal analysis are two basic important techniques used in finding solutions for a network. The suitability of either mesh or nodal analysis to a particular problem depends mainly on the number of voltage sources or current sources. If a network has a large number of voltage sources, it is useful to use mesh analysis; as this analysis requires that all the sources in a circuit be voltage sources. Therefore, if there are any current sources in a circuit they are to be converted into equivalent voltage sources, if, on the other hand, the network has more current sources, nodal analysis is more useful.

1. SUPERPOSITION THEOREM

Statement:

The superposition theorem states that in any linear network containing two or more sources, the response in any element is equal to the algebraic sum of the responses caused by individual sources acting alone, while the other sources are non-operative; that is, while considering the effect of individual sources, other ideal voltage sources and ideal current sources in the network are replaced by short circuit and open circuit across their terminals respectively. This theorem is valid only for linear systems.

Procedure:

Step 1 – Identify the number of sources in the given network.

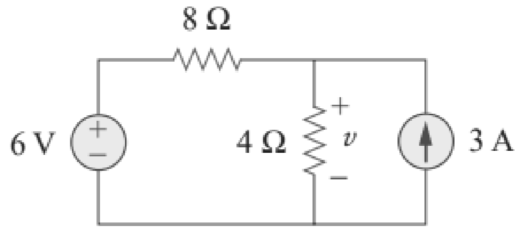
Step 2– Find the response (finding value) in a particular branch by considering one independent source and eliminating the remaining independent sources present in the network. [The dependent sources are not removed.]

Step 3 – Repeat Step 1 for all other independent sources present in the network.

Step 4 – Add all the responses in order to get the overall response (required value) in a particular branch when all independent sources are present in the network.

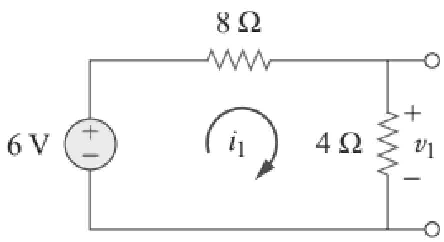
PROBLEMS ON SUPERPOSITION THEOREM

1) Determine the voltage ‘v’ in the circuit using Superposition theorem.



SOL:

Step-1 : 6V source is acting alone. 3A source is open circuited. The modified circuit is shown in the fig.

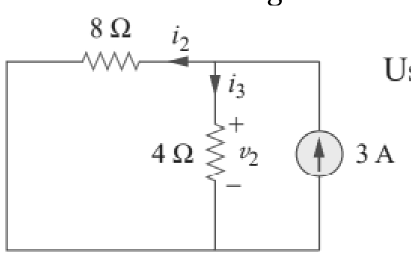


Applying KVL to the loop

$$12i_1 - 6 = 0 \quad \Rightarrow \quad i_1 = 0.5 \text{ A}$$

$$v_1 = 4i_1 = 2 \text{ V}$$

Step-2 : 3A source is acting alone. 6V source is short circuited.



Using current division,

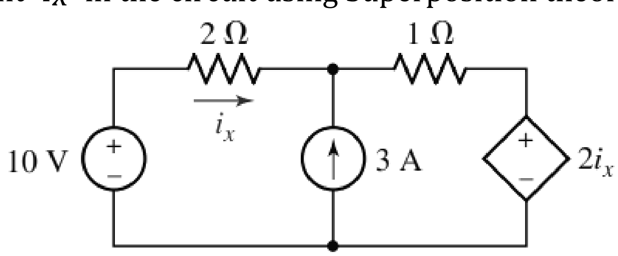
$$i_3 = \frac{8}{4 + 8}(3) = 2 \text{ A}$$

Hence, $v_2 = 4i_3 = 8$ V

Step-3: The voltage ‘v’ is

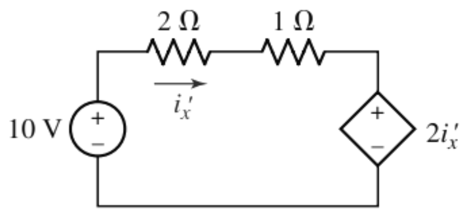
$$v = v_1 + v_2 = 2 + 8 = 10 \text{ V}$$

2) Determine the current ‘i_x’ in the circuit using Superposition theorem.



SOL:

Step-1 : 10V source is acting alone. 3A source is open circuited.

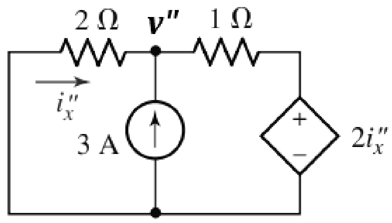


Apply mesh analysis

$$-10 + 2i_x' + i_x' + 2i_x' = 0$$

$$i_x' = 2\text{ A}$$

Step-2 : 3A source is acting alone. 10V source is short circuited.



Apply Nodal analysis,

$$\frac{v''}{2} + \frac{v'' - 2i_x''}{1} = 3$$

relate the dependent-source-controlling quantity to v'':

$$v'' = 2(-i_x'')$$

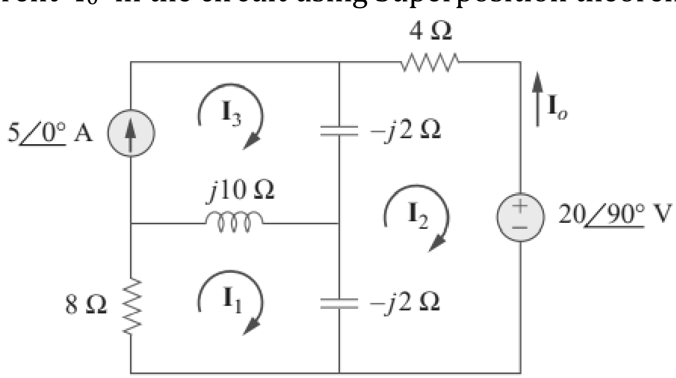
By solving

$$i_x'' = -0.6\text{ A}$$

Step-3 : The current 'i_x' is

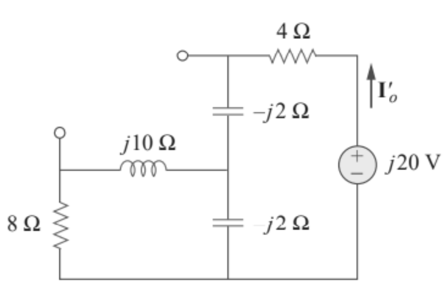
$$i_x = i_x' + i_x'' = 2 + (-0.6) = 1.4\text{ A}$$

3) Determine the current 'I_0' in the circuit using Superposition theorem.



SOL:

Step-1 : $20\angle 90^0$ is acting alone. 5A source is open circuited.



If we let $\mathbf{Z}$ be the parallel combination of $-j2$ and $8 + j10$, then

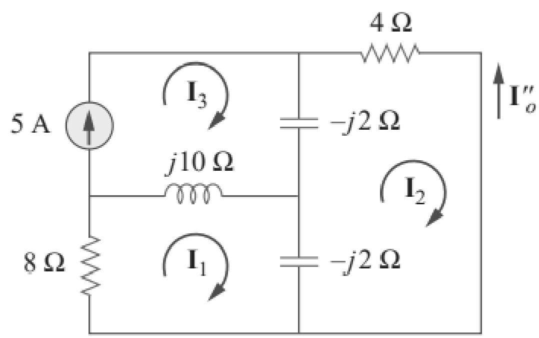
$$\mathbf{Z} = \frac{-j2(8 + j10)}{-2j + 8 + j10} = 0.25 - j2.25$$

and current $\mathbf{I}_o'$ is

$$\mathbf{I}_o' = \frac{j20}{4 - j2 + \mathbf{Z}} = \frac{j20}{4.25 - j4.25}$$

$$\mathbf{I}_o' = -2.353 + j2.353$$

Step-2: 5A source is acting alone. 20V source is short circuited.



For mesh 1, $(8 + j8)\mathbf{I}_1 - j10\mathbf{I}_3 + j2\mathbf{I}_2 = 0$

For mesh 2, $(4 - j4)\mathbf{I}_2 + j2\mathbf{I}_1 + j2\mathbf{I}_3 = 0$

For mesh 3, $\mathbf{I}_3 = 5$

Solving the equations

$$(4 - j4)\mathbf{I}_2 + j2\mathbf{I}_1 + j10 = 0$$

Expressing $\mathbf{I}_1$ in terms of $\mathbf{I}_2$ gives

$$\mathbf{I}_1 = (2 + j2)\mathbf{I}_2 - 5$$

Substitute I_1 and I_3 in first equation

$$(8 + j8)[(2 + j2)\mathbf{I}_2 - 5] - j50 + j2\mathbf{I}_2 = 0$$

$$\mathbf{I}_2 = \frac{90 - j40}{34} = 2.647 - j1.176$$

Current $\mathbf{I}_o''$ is obtained as

$$\mathbf{I}_o'' = -\mathbf{I}_2 = -2.647 + j1.176$$

Step-3 : The current I_0 is

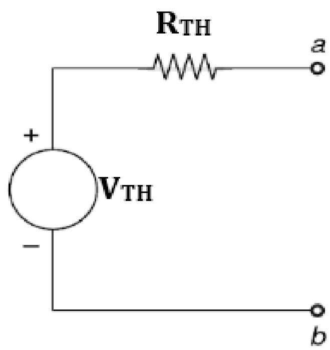
$$\mathbf{I}_o = \mathbf{I}_o' + \mathbf{I}_o'' = -5 + j3.529 = 6.12\underline{/144.78^\circ}\text{ A}$$

2. THEVENIN'S THEOREM

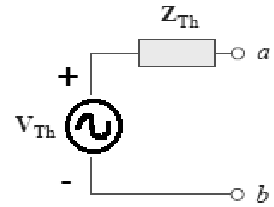
Statement:

Thevenin's theorem states that any two terminal linear networks having a number of voltage-current sources and resistances(impedances) can be replaced by a simple equivalent circuit consisting of a single voltage source in series with a resistance(impedance), where the value of the voltage source is equal to the open circuit voltage across the two terminals of the network, and resistance is equal to the equivalent resistance measured between the terminals with all the energy sources are replaced by their internal resistances.

According to Thevenin's theorem, The Thevenin's equivalent circuit can be shown in the following circuit.

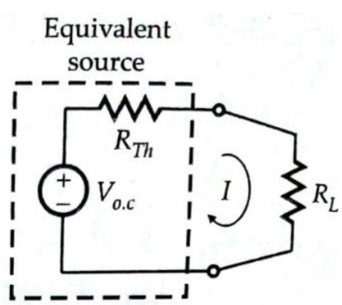


(or)



Steps for Solution of a Network Utilizing Thevenin's Theorem

- Remove the load resistor (R_L) and find the open circuit voltage (V_{oc} or V_{Th}) across the open circuited load terminals.
- Deactivate the constant sources (for voltage source, remove it by internal resistance and for current source delete the source by open circuit) and find the internal resistance (Thevenin's resistance) of the source side looking through the open circuited load terminals. Let this resistance be R_{Th}.
- Obtain Thevenin's equivalent circuit by placing R_{Th} in series with V_{oc}.
- Reconnect R_L across the load terminals as shown in Fig.

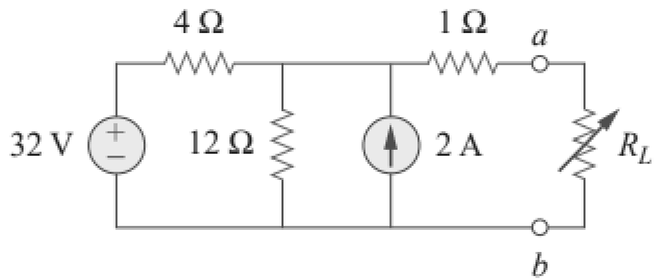


Obviously *I* (the load current)

$$= \frac{V_{o.c}}{R_{Th} + R_L}.$$

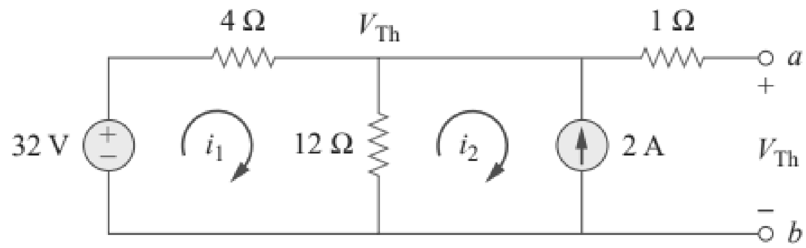
PROBLEMS ON THEVENIN'S THEOREM

4) Determine the Thevenin's equivalent circuit across the load resistance R_L.



SOL:

Step-1: Open R_L and find the Thevenin's voltage V_{Th} across R_L.



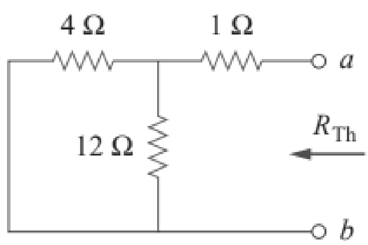
Applying mesh analysis to the two loops, we obtain

$$-32 + 4i_1 + 12(i_1 - i_2) = 0, \qquad i_2 = -2\text{ A}$$

Solving for i_1, we get $i_1 = 0.5$ A. Thus,

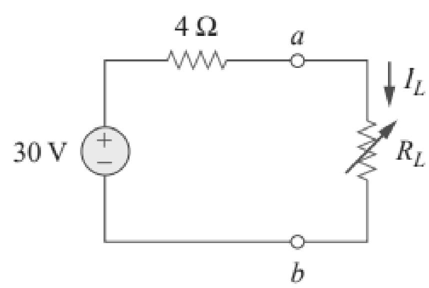
$$V_{Th} = 12(i_1 - i_2) = 12(0.5 + 2.0) = 30\text{ V}$$

Step-2: To find the Thevenin's resistance R_{Th}.

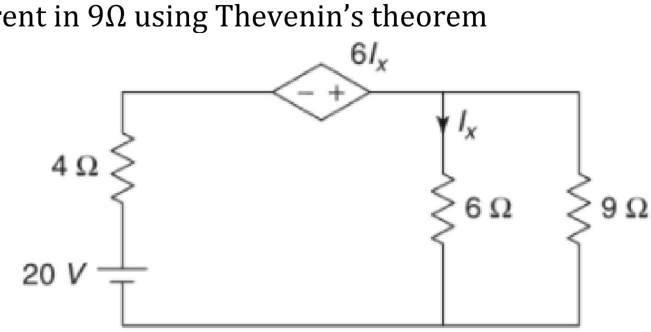


$$R_{Th} = 4 \parallel 12 + 1 = \frac{4 \times 12}{16} + 1 = 4\ \Omega$$

Step-3 : The Thevenin's equivalent circuit is shown in the fig.



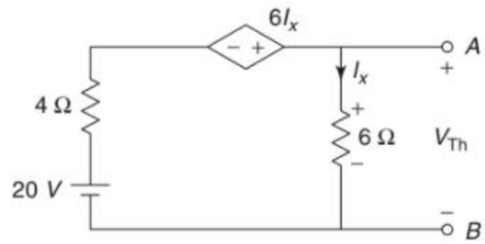
5) Determine the current in 9Ω using Thevenin's theorem



SOL:

Step I Calculation of V_{Th}
Applying KVL to the mesh,

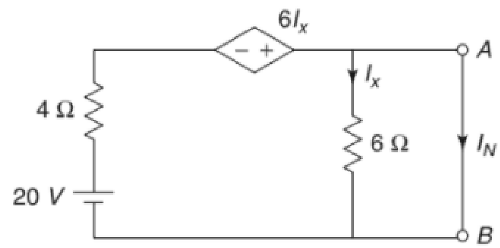
$$20 - 4I_x + 6I_x - 6I_x = 0$$
$$I_x = 5\text{ A}$$



Writing the V_{Th} equation,

$$6I_x - V_{Th} = 0$$
$$6(5) - V_{Th} = 0$$
$$V_{Th} = 30\text{ V}$$

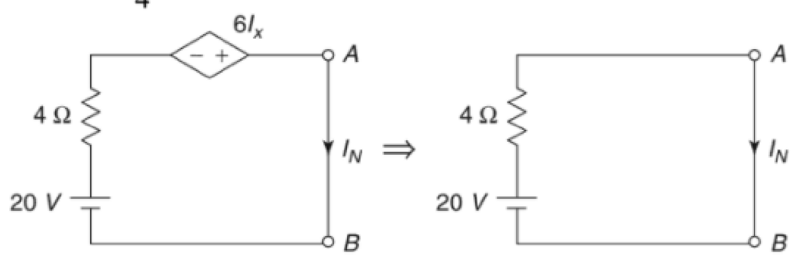
Step II Calculation of I_N
From Fig.



$$I_x = 0$$

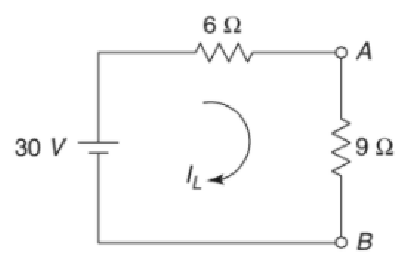
The dependent source $6I_x$ depends on the controlling variable I_x. When $I_x = 0$, the dependent source vanishes, i.e., $6I_x = 0$

$$I_N = \frac{20}{4} = 5\text{ A}$$



Step III Calculation of R_{Th}

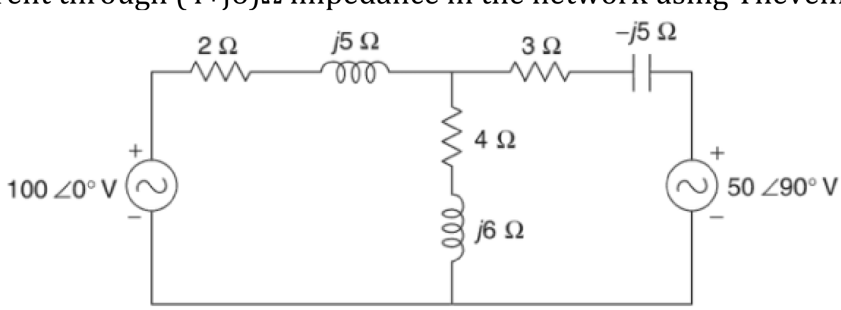
$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{30}{5} = 6\ \Omega$$



Step IV Calculation of I_L

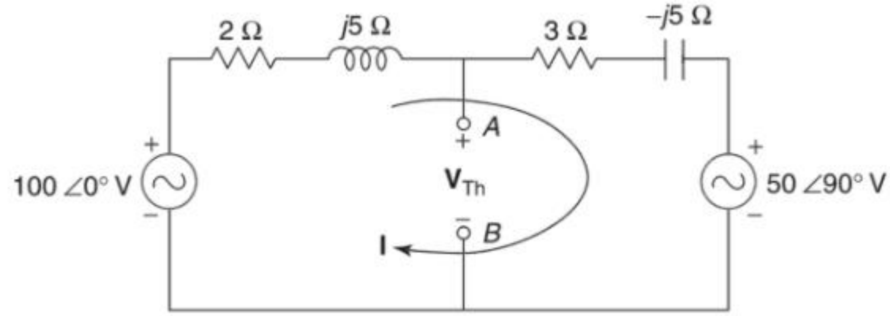
$$I_L = \frac{30}{6+9} = 2\text{ A}$$

6) Find the current through (4+j6)Ω impedance in the network using Thevenin's theorem.



SOL:

Step I Calculation of $\mathbf{V}_{\text{Th}}$



Applying KVL to the mesh,

$$100\angle 0° - 2\mathbf{I} - j5\mathbf{I} - 3\mathbf{I} + j5\mathbf{I} - 50\angle 90° = 0$$

$$\mathbf{I} = 22.36\angle -26.57° \text{ A}$$

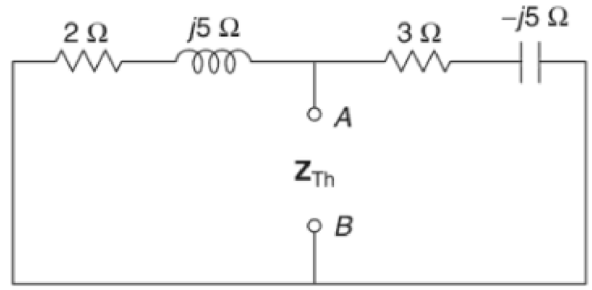
Writing $\mathbf{V}_{\text{Th}}$ equation,

$$\mathbf{V}_{\text{Th}} - 3\mathbf{I} + j5\mathbf{I} - 50\angle 90° = 0$$

$$\mathbf{V}_{\text{Th}} - (3 - j5)(22.36\angle -26.57°) - 50\angle 90° = 0$$

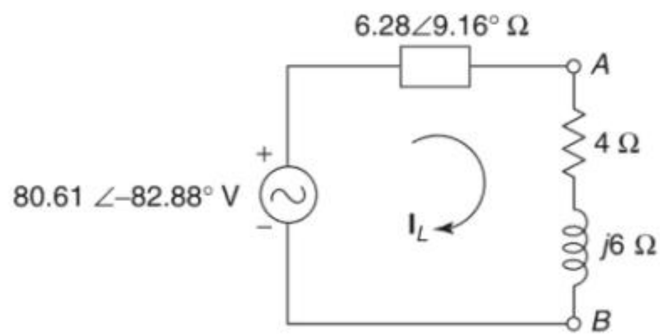
$$\mathbf{V}_{\text{Th}} = 80.61\angle -82.88° \text{ V}$$

Step II Calculation of $\mathbf{Z}_{\text{Th}}$



$$\mathbf{Z}_{\text{Th}} = \frac{(2 + j5)(3 - j5)}{2 + j5 + 3 - j5} = 6.28\angle 9.16° \ \Omega$$

Step III Calculation of $\mathbf{I}_L$



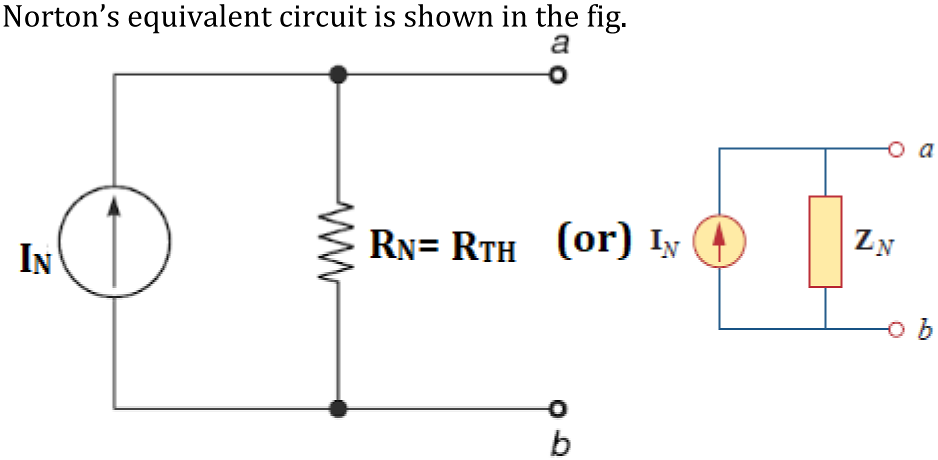
$$\mathbf{I}_L = \frac{80.61\angle -82.88°}{6.28\angle 9.16° + 4 + j6} = 6.52\angle -117.34° \text{ A}$$

3. NORTON'S THEOREM

Statement:

Norton's theorem states that any two terminal linear network with voltage-current sources and resistances (impedances) can be replaced by an equivalent circuit consisting of a current source in parallel with a resistance (impedance). The value of current source is the short circuit current(Norton's current) between the two terminals of the network and resistance (Norton's resistance or impedance) is the equivalent resistance (impedance) measured between the terminals of the network with all energy sources are replaced by their internal resistances.

The Norton's equivalent circuit is shown in the fig.



It is easily seen that the Norton equivalent follows from the Thevenin equivalent by source conversion and also vice versa.

In the Norton equivalent we will write

$R_0 = R_N$, the Norton resistance

Obviously

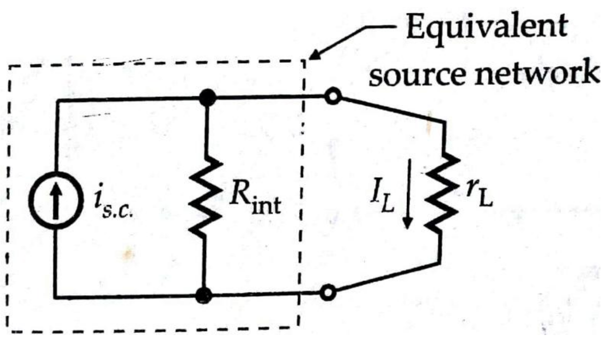
$$R_N = R_{TH}$$

From the source conversion

$$I_{SC} = \frac{V_{OC}}{R_0} = \frac{V_{TH}}{R_{TH}}$$

Steps for Solution of a Network Utilizing Norton's Theorem

- Remove the load resistor and find the internal resistance of the source network by deactivating the constant sources. This procedure is exactly same as described for Thevenin's theorem. Let this resistance be R_{int} or R_N.
- Next, short the load terminals and find the short circuit current flowing through the shorted load terminals using conventional network analysis. Let this current be i_{SC} or i_N.
- Norton's equivalent circuit is drawn by keeping R_{int} in parallel to i_{SC} as shown in Fig.

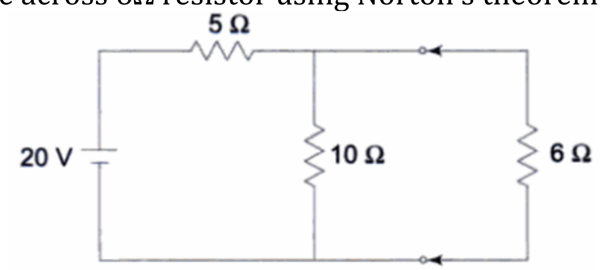


- Reconnect the load resistor (RL) across the load terminals and the current through it (I_L) is then given by

$$I_L = i_{s.c}\frac{R_{int}}{R_{int} + R_L}$$

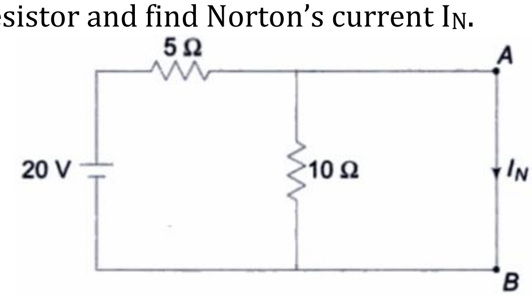
PROBLEMS ON NORTON'S THEOREM

7) Determine the voltage across 6Ω resistor using Norton's theorem in the following circuit.



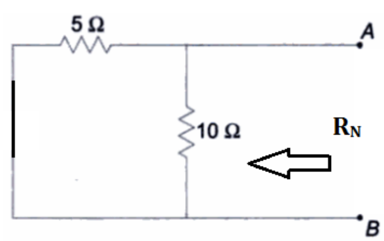
SOL:

Step-1: Short circuit 6Ω resistor and find Norton's current I_N.



$$I_N = \frac{20}{5} = 4\ \text{A}$$

Step-2 : Find Norton's resistance R_N.



Norton's resistance is equal to the parallel combination of both the 5 Ω and 10 Ω resistors

$$R_N = \frac{5 \times 10}{15} = 3.33\ \Omega$$

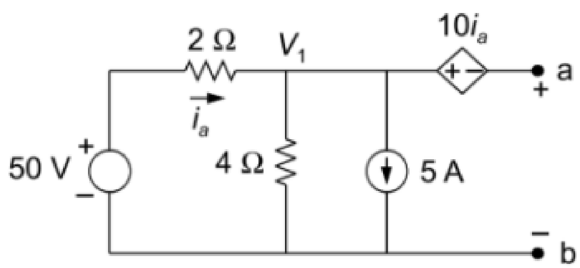
Step-3: Draw the Norton's equivalent circuit and find voltage across 6Ω.

Now let us find the current passing through the 6 Ω resistor and the voltage across it due to Norton's equivalent circuit.

$$I_6 = 4 \times \frac{3.33}{6 + 3.33} = 1.43 \text{ A}$$

The voltage across the 6 Ω resistor = 1.43 × 6 = 8.58 V

8) Find the Thevenin's and Norton's equivalent circuits across the terminals a and b in the circuit.



SOL:

Step I: Determination of V_{Th} or V_{OC}

To find V_{Th}, first open terminals ab as shown in Figure and find V_{OC}. Writing KCL at the only node, we have

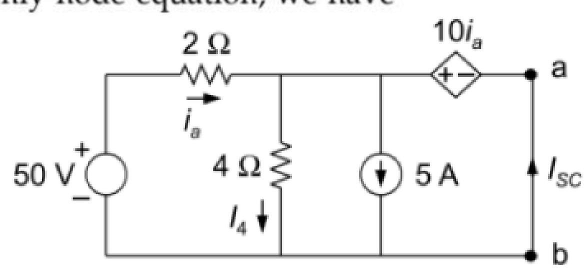
$$\frac{V_1 - 50}{2} + \frac{V_1}{4} + 5 = 0 \quad \text{or} \quad 3V_1 = 80 \quad \text{or} \quad V_1 = 26.666 \text{ V}$$

$$i_a = \frac{50 - V_1}{2} = \frac{50 - 26.66}{2} = 11.67 \text{ A}$$

$$V_{Th} = V_{OC} = V_1 - 10i_a = 26.666 - 10 \times 11.67 = -90.034 \text{ V} \approx -90 \text{ V}$$

Step II: Determination of I_{SC}

To find I_{SC}, short terminals ab as shown in Figure Now, the voltage at the only node is $V_1 = 10i_a$. Writing the only node equation, we have



$$\frac{10i_a}{4}+5+\frac{10i_a-50}{2}=0 \quad \text{or} \quad 30i_a=80 \quad \text{or} \quad i_a=2.666\text{ A}$$

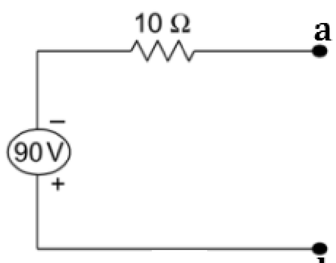
$$I_4 = \frac{10i_a}{4} = \frac{10\times 2.66}{4} = 6.66\text{ A}$$

$$I_{SC} = I_4 + 5 - i_a = 6.66+5-2.66 = 9\text{ A}$$

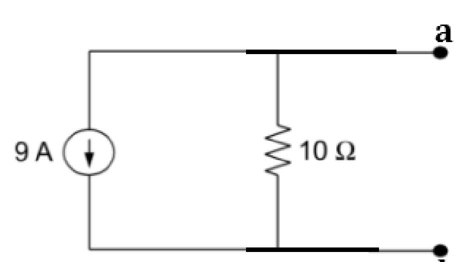
Step III: Determination of R_{Th}

$$R_{Th} = \frac{V_{OC}}{I_{SC}} = \frac{90\text{ V}}{9\text{ A}} = 10\ \Omega$$

So the Thevenin's and Norton's equivalent circuits are as shown in Figure respectively.

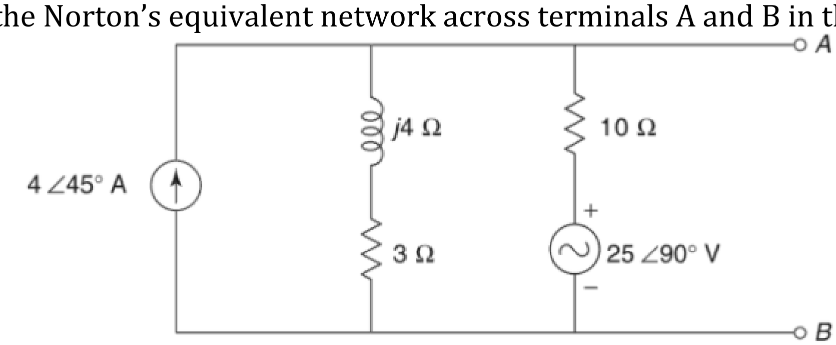


Thevenin equivalent circuit



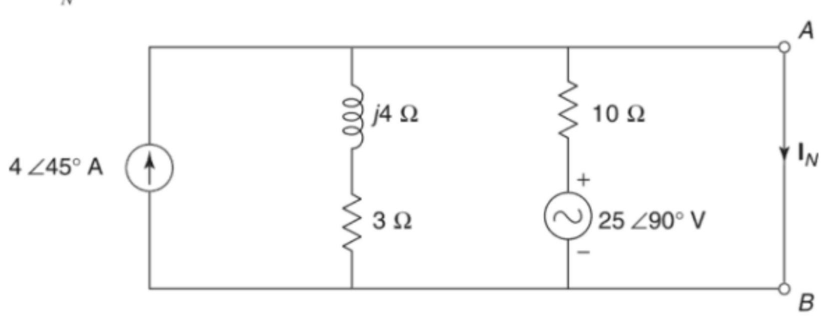
Norton equivalent circuit

9) Determine the Norton's equivalent network across terminals A and B in the circuit

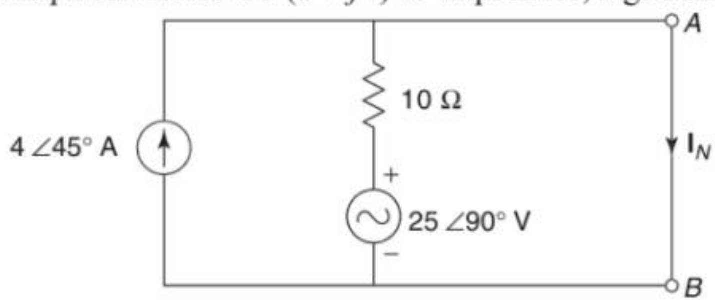


SOL:

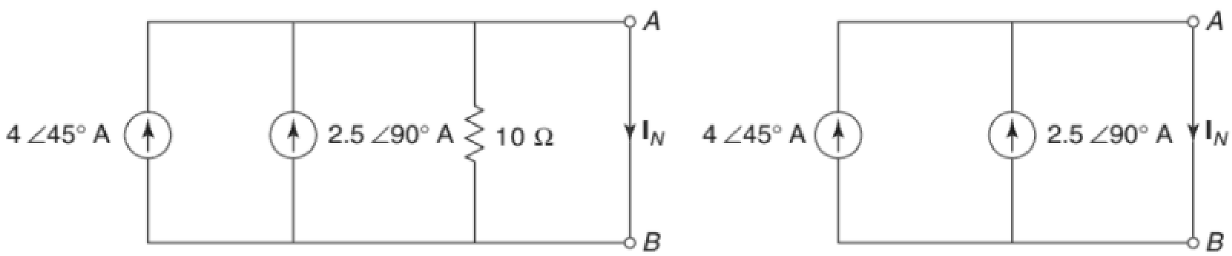
Step I Calculation of $\mathbf{I}_N$



When a short circuit is placed across the $(3+j4)\ \Omega$ impedance, it gets shorted

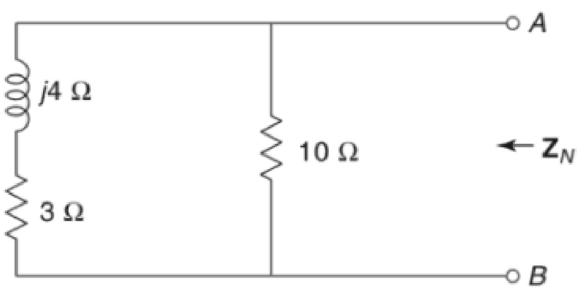


By source transformation, the network is redrawn as shown in Fig



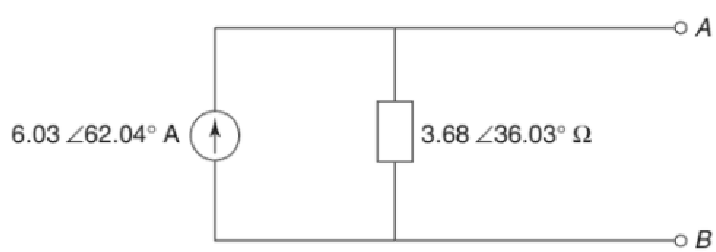
$$I_N = 4\angle 45^\circ + 2.5\angle 90^\circ = 6.03\angle 62.04^\circ \text{ A}$$

Step II Calculation of $\mathbf{Z}_N$



$$\mathbf{Z}_N = \frac{10(3+j4)}{10+3+j4} = 3.68\angle 36.03^\circ\ \Omega$$

Step III Norton's Equivalent Network

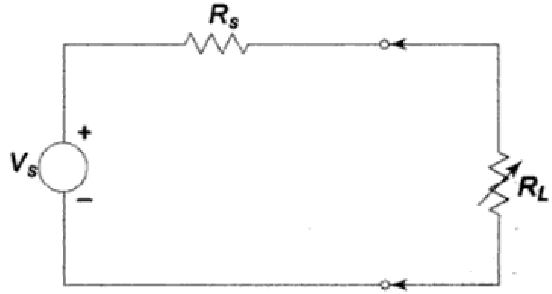


4. MAXIMUM POWER TRANSFER THEOREM

Statement:

The maximum Power Transfer Theorem states that maximum power is delivered from a source to a load when the load resistance is equal to the source resistance.

In Fig., assume that the load resistance is variable.



Current in the circuit is $I = V_S/(R_S + R_L)$

Power delivered to the load R_L is $P = I^2R_L = V_S^2R_L/(R_S + R_L)^2$

To determine the value of R_L for maximum power to be transferred to the load, we have to set the first derivative of the above equation with respect to R_L, i.e. when $\frac{dP}{dR_L}$ equals zero.

$$\frac{dP}{dR_L} = \frac{d}{dR_L}\left[\frac{V_S^2}{(R_S + R_L)^2} R_L\right]$$

$$= \frac{V_S^2\left\{(R_S + R_L)^2 - (2R_L)(R_S + R_L)\right\}}{(R_S + R_L)^4}$$

$$(R_S + R_L)^2 - 2R_L(R_S + R_L) = 0$$

$$R_S^2 + R_L^2 + 2R_SR_L - 2R_L^2 - 2R_SR_L = 0$$

$$R_S = R_L$$

So, maximum power will be transferred to the load when load resistance is equal to the source resistance.

The maximum power is

$$P_{max} = I^2 R_L = \frac{V_{TH}^2 R_L}{(2R_L)^2} \qquad \ldots \text{ as } I = \frac{V_{TH}}{2R_L}$$

$$\therefore \qquad \boxed{P_{max} = \frac{V_{TH}^2}{4R_L}}$$

when V_{TH} = Thevenin's voltage as circuit is replaced by its Thevenin's equivalent

But the efficiency of the circuit is 50 %, when $P = P_{max}$.

For AC circuits:

maximum power transfer occurs when $Z_L = R_L + jX_L = R_S - jX_S = Z_S^*$

That is, load impedance is equal to the complex conjugate of the source impedance. That is, $Z_L = Z_S^*$

Proof:

$$Z_{th} = R_{th} + jX_{th} \qquad Z_L = R_L + jX_L$$

$$I_L = \frac{V_{th}}{Z_{th} + Z_L}$$

$$I_L = \frac{V_{th}}{(R_{th} + R_L) + j(X_{th} + X_L)}$$

$$I_L = \frac{V_{th}}{\sqrt{(R_{th} + R_L)^2 + (X_{th} + X_L)^2}}$$

Power Transferred to the load is

$$P_L = I_L^2 R_L = \frac{V_{th}^2 R_L}{(R_{th} + R_L)^2 + (X_{th} + X_L)^2} \text{--------}(1)$$

For any network V_{th}, R_{th} and X_{th} are fixed.

CASE 1: When R_L is constant and X_L is varying:

$$\frac{dP_L}{dX_L} = \frac{d}{dX_L}\left[\frac{V_{th}^2 R_L}{(R_{th} + R_L)^2 + (X_{th} + X_L)^2}\right] = 0$$

$$\frac{dP_L}{dX_L} = \frac{(R_{th} + R_L)^2 + (X_{th} + X_L)^2 [0] - V_{th}^2 R_L [2(X_{th} + X_L]}{[(R_{th} + R_L)^2 + (X_{th} + X_L)^2]^2} = 0$$

$$- V_{th}^2 R_L [2(X_{th} + X_{th}] = 0$$

$$\boxed{X_L = - X_{th}} \text{-----------}(2)$$

Hence, the magnitude of load reactance must be equal Thevenin's reactance but opposite phase difference. This is the first condition. The equation (1) becomes

$$P_L = \frac{V_{th}^2 R_L}{(R_{th} + R_L)^2} \text{-----}(3)$$

CASE 2: When X_L is constant and R_L is varying

$$\frac{dP_L}{dR_L} = \frac{d}{dR_L}\left[\frac{V_{th}^2 R_L}{(R_{th}+R_L)^2 + (X_{th}+X_L)^2}\right] = 0$$

$$\frac{dP_L}{dX_L} = \frac{(R_{th}+R_L)^2 + (X_{th}+X_L)^2 [V_{th}^2] - V_{th}^2 R_L [2(R_{th}+R_L)]}{\left[(R_{th}+R_L)^2 + (X_{th}+X_L)^2\right]^2} = 0$$

$$(R_{th}+R_L)^2 + (X_{th}+X_L)^2 [V_{th}^2] - V_{th}^2 R_L [2(R_{th}+R_L)] = 0$$

$$R_{th}^2 + (X_{th}+X_L)^2 = R_L \text{ -------(4)}$$

Put $X_{th} = -X_L$ in equation (4)

$\boxed{R_L = R_{th}}$ --------------- (5)

Hence, for maximum Power Transfer, the load resistance must be equal to the Thevenin's Equivalent Resistance. The equation (3) becomes

$$P_{L\,max} = \frac{V_{th}^2}{4R_{th}} \text{ -----(6)}$$

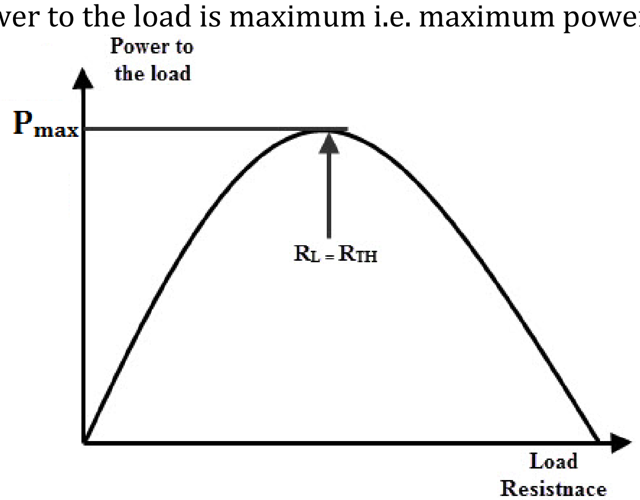
CASE 3: Apply equation (2) and (5) simultaneously then

$$Z_L = R_L + jX_L = R_{th} - jX_{th} = Z_{th}^*$$

maximum power transfer occurs when $Z_L = R_L + jX_L = R_S - jX_S = Z_S^*$

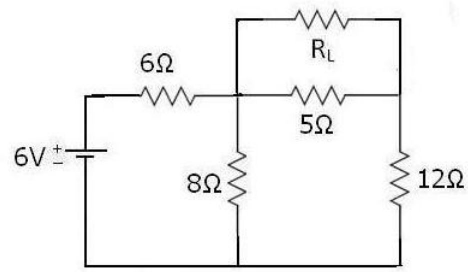
That is, load impedance is equal to the complex conjugate of the source impedance. That is, $Z_L = Z_S^*$

The variation of power to the load with respect to the load resistance is shown in the fig. When $R_L = R_{Th}$, the power to the load is maximum i.e. maximum power is delivered to the load.



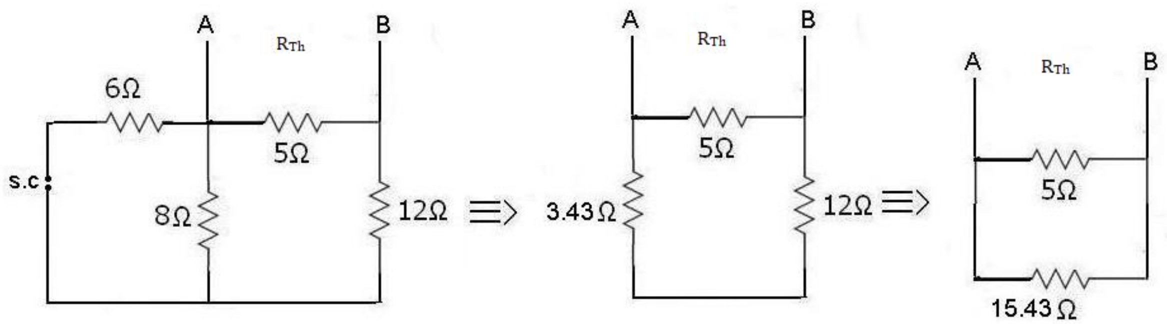
PROBLEMS ON MAXIMUM POWER TRANSFER THEOREM

10) Find the value of R_L for the given network below that the power is maximum And also find the Max Power through load-resistance R_L



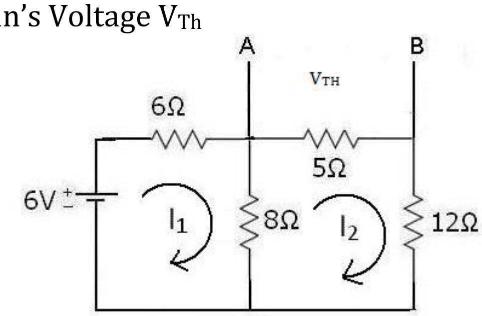
SOL:

Step-1: To find the unknown resistance R_L by short circuiting 6V source



$$\mathbf{R_{Th} = 5 \,||\, 15.43 = 3.77\ \Omega}$$

Step-2: To find the Thevenin's Voltage V_{Th}



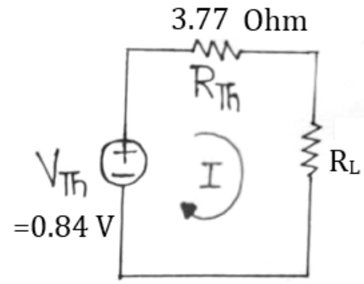
Apply KVL to the loops,

$$14I_1 - 8I_2 = 6$$
$$-8I_1 + 25I_2 = 0$$

Solving above equations,

$$I_1 = 0.524\text{ A},\quad I_2 = 0.168\text{ A}$$
$$\mathbf{V_{Th} = 5I_2 = 5 \times 0.168 = 0.84\ V}$$

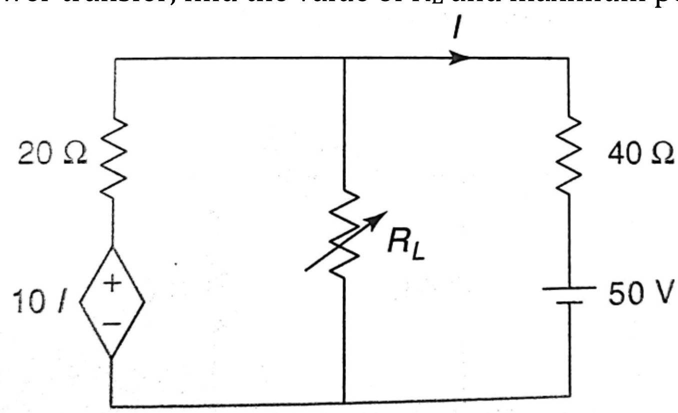
Step-3 : To draw the thevenin's equivalent circuit



From maximum power transfer theorem, **R_L = 3.77 Ω**

$$\text{Maximum power, } P_{max} = \frac{{V_{Th}}^2}{4R_{Th}} = \frac{0.84^2}{4\times 3.77} = 0.046 \text{ W}$$

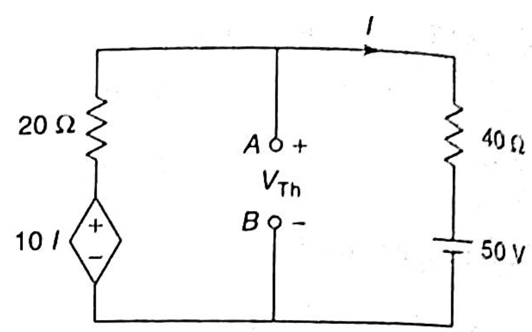
11) For maximum power transfer, find the value of R_L and maximum power in the circuit



SOL:

Step I Calculation of V_{Th}

Applying KVL to the mesh,



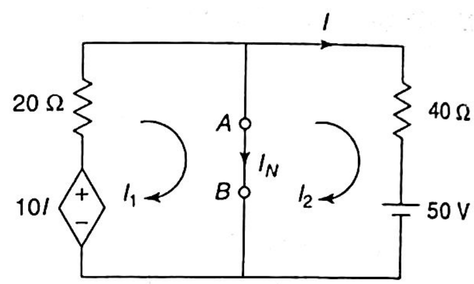
$$10I - 20I - 40I - 50 = 0$$
$$I = -1 \text{ A}$$

Writing the V_{Th} equation,

$$V_{Th} - 40I - 50 = 0$$
$$V_{Th} - 40(-1) - 50 = 0$$
$$V_{Th} = 10 \text{ V}$$

Step II Calculation of I_N

From Fig.



$$I = I_2 \qquad \ldots\text{(i)}$$

Applying KVL to Mesh 1,

$$10I - 20I_1 = 0$$
$$10I_2 - 20I_1 = 0 \qquad \ldots\text{(ii)}$$

Applying KVL to Mesh 2,

$$-40I_2 - 50 = 0$$

$$I_2 = -1.25 \text{ A} \quad \text{...(iii)}$$

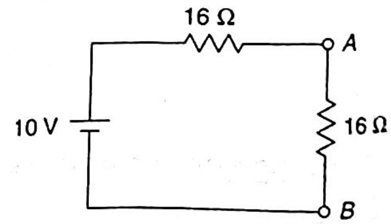
Solving Eqs (i), (ii) and (iii),

$$I_1 = -0.625 \text{ A}$$

$$I_N = I_1 - I_2 = -0.625 + 1.25 = 0.625 \text{ A}$$

Step III Calculation of R_N

$$R_{Th} = \frac{V_{Th}}{I_N} = \frac{10}{0.625} = 16\ \Omega$$



Step IV Calculation of R_L

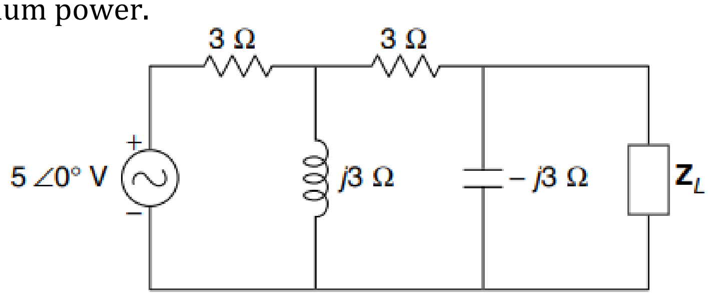
For maximum power transfer,

$$R_L = R_{Th} = 16\ \Omega$$

Step V Calculation of P_{max}

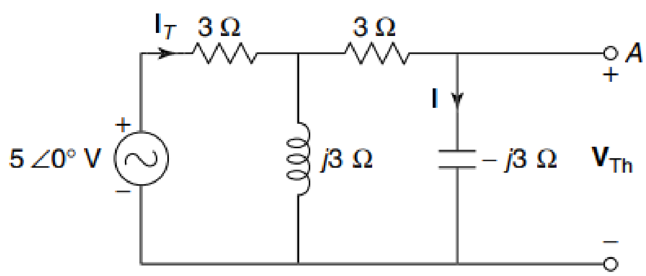
$$P_{max} = \frac{V_{Th}}{4R_{Th}} = \frac{(10)^2}{4\times16} = 1.56 \text{ W}$$

12) Find the impedance ZL so that maximum power can be transferred to it in the network. Calculate the maximum power.



SOL:

Step I Calculation of $\mathbf{V}_{Th}$



$$\mathbf{Z}_T = 3 + \frac{j3(3-j3)}{3+j3-j3} = 6.71\angle 26.57^\circ\ \Omega$$

$$\mathbf{I}_T = \frac{5\angle 0^\circ}{6.71\angle 26.57^\circ} = 0.75\angle -26.57^\circ \text{ A}$$

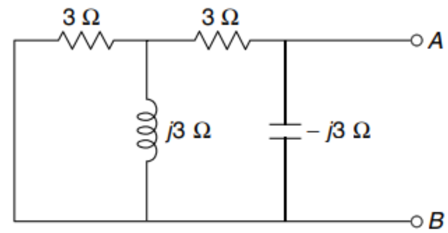
By current division rule,

$$\mathbf{I} = 0.75\angle -26.57^\circ \times \frac{j3}{3+j3-j3} = 0.75\angle 63.43^\circ \text{ A}$$

$$\mathbf{V}_{Th} = (-j3)(0.75\angle 63.43^\circ) = 2.24\angle -26.57^\circ \text{ V}$$

Step II Calculation of $\mathbf{Z}_{Th}$

$$\mathbf{Z}_{Th} = [(3 \parallel j3) + 3] \parallel (-j3)$$
$$= 3 \angle -53.12° \ \Omega$$
$$= (1.8 - j2.4) \ \Omega$$

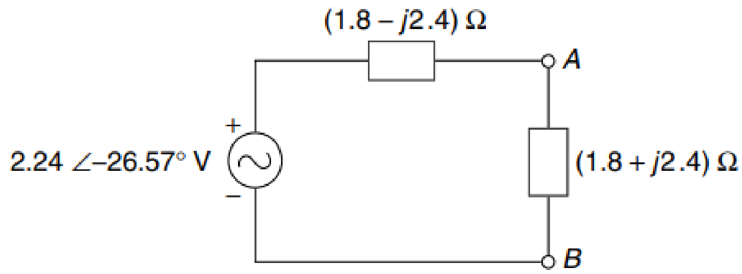


Step III Calculation of $\mathbf{Z}_L$

For maximum power transfer, the load impedance should be a complex conjugate of the source impedance.

$$\mathbf{Z}_L = (1.8 + j2.4) \ \Omega$$

Step IV Calculation of P_{max}

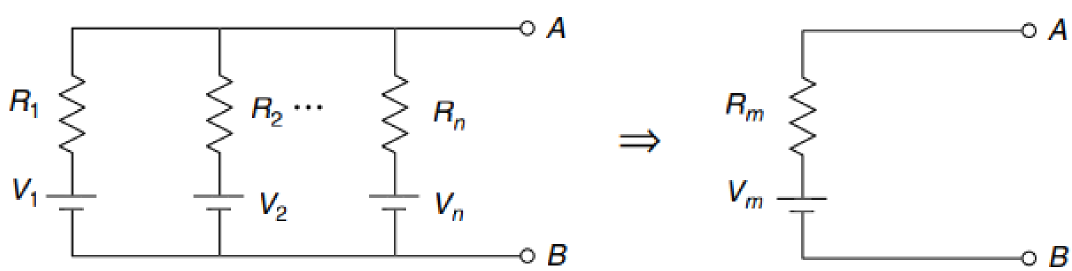


$$P_{max} = \frac{|\mathbf{V}_{Th}|^2}{4R_L} = \frac{|2.24|^2}{4 \times 1.8} = 0.7 \text{ W}$$

5. <u>MILLMAN'S THEOREM</u>

Statement:

It states that 'if there are n voltage sources V_1, V_2 ,…,V_n with internal resistances R_1,R_2 ,…, R_n respectively connected in parallel then these voltage sources can be replaced by a single voltage source V_m and a single series resistance R_m' as shown in the fig.



where
$$V_m = \frac{V_1G_1 + V_2G_2 + \ldots + V_nG_n}{G_1 + G_2 + \ldots + G_n}$$
$$R_m = \frac{1}{G_m} = \frac{1}{G_1 + G_2 + \ldots + G_n}$$

For AC circuit:

Millman's theorem states that 'If there are n voltage sources V_1, V_2, ... Vn with internal impedances Z_1, Z_2, ... Zn respectively connected in parallel then these voltage sources can be replaced by a single voltage source V_m and a single series impedance Z_m.

$$\mathbf{V}_m = \frac{\mathbf{V}_1\ \mathbf{Y}_1 + \mathbf{V}_2\ \mathbf{Y}_2 + \ldots + \mathbf{V}_n\ \mathbf{Y}_n}{\mathbf{Y}_1 + \mathbf{Y}_2 + \ldots + \mathbf{Y}_n}$$

$$\mathbf{Z}_m = \frac{1}{\mathbf{Y}_m} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2 + \ldots + \mathbf{Y}_n}$$

Steps to be followed in Millman's Theorem

1. Remove the load resistance R_L.
2. Find Millman's voltage across points A and B.

$$V_m = \frac{V_1 G_1 + V_2 G_2 + \ldots + V_n G_n}{G_1 + G_2 + \ldots + G_n}$$

3. Find the resistance R_m between points A and B.

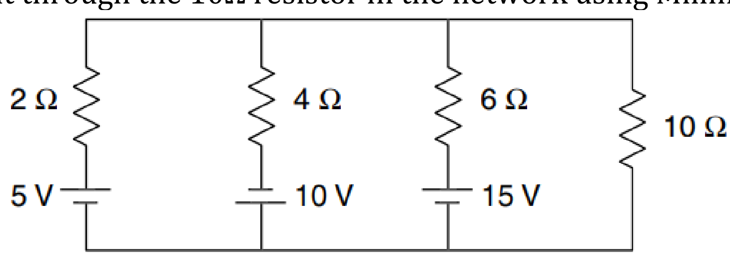
$$R_m = \frac{1}{G_1 + G_2 + \ldots + G_n}$$

4. Replace the network by a voltage source V_m in series with the resistance R_m.
5. Find the current through R_L using ohm's law.

$$I_L = \frac{V_m}{R_m + R_L}$$

PROBLEMS ON MILLIMAN THEOREM

13) Find the current through the 10Ω resistor in the network using Millman's theorem.



SOL:

Step I Calculation of V_m

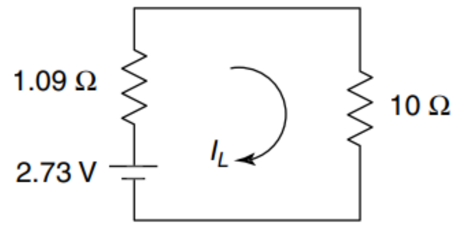
$$V_m = \frac{V_1 G_1 + V_2 G_2 + V_3 G_3}{G_1 + G_2 + G_3} = \frac{5\left(\frac{1}{2}\right) - 10\left(\frac{1}{4}\right) + 15\left(\frac{1}{6}\right)}{\frac{1}{2} + \frac{1}{4} + \frac{1}{6}} = 2.73 \text{ V}$$

Step II Calculation of R_m

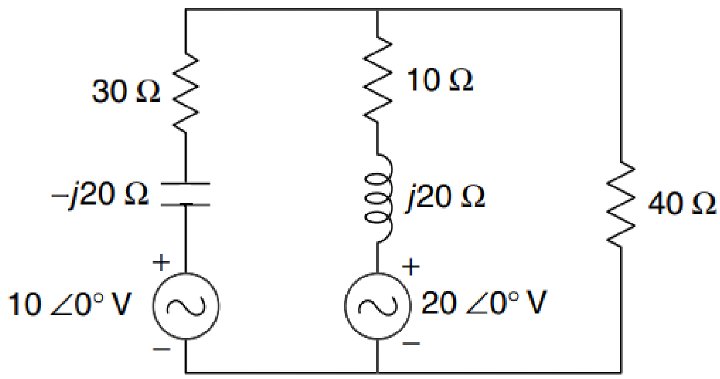
$$R_m = \frac{1}{G_m} = \frac{1}{\frac{1}{2}+\frac{1}{4}+\frac{1}{6}} = 1.09\ \Omega$$

Step III Calculation of I_L

$$I_L = \frac{2.73}{1.09+10} = 0.25\ \text{A}$$



14) Find the current through the 40 Ω resistor for the network using Millman's theorem.



SOL:

Step I Calculation of $\mathbf{V}_m$

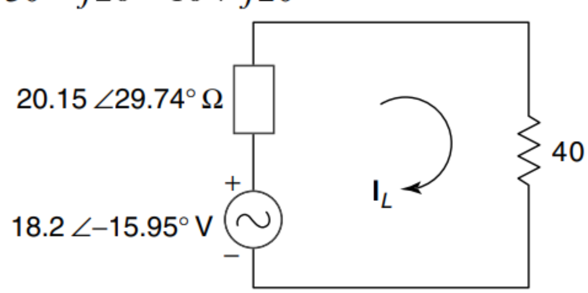
$$\mathbf{V}_m = \frac{\mathbf{V}_1\mathbf{Y}_1+\mathbf{V}_2\mathbf{Y}_2}{\mathbf{Y}_1+\mathbf{Y}_2} = \frac{(10\angle 0^\circ)\left(\frac{1}{30-j20}\right)+(20\angle 0^\circ)\left(\frac{1}{10+j20}\right)}{\frac{1}{30-j20}+\frac{1}{10+j20}} = 18.2\angle -15.95^\circ \text{V}$$

Step II Calculation of $\mathbf{Z}_m$

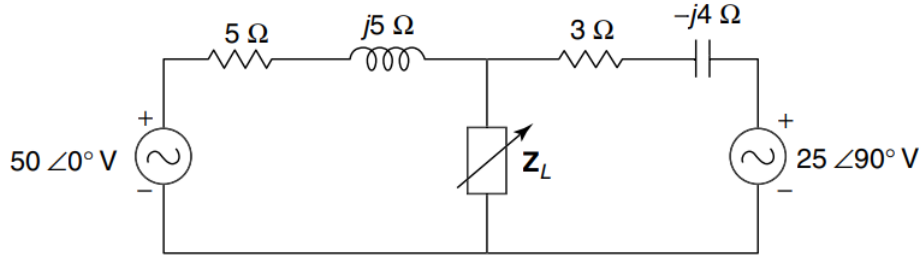
$$\mathbf{Z}_m = \frac{1}{\mathbf{Y}_m} = \frac{1}{\mathbf{Y}_1+\mathbf{Y}_2} = \frac{1}{\frac{1}{30-j20}+\frac{1}{10+j20}} = 20.15\angle 29.74^\circ\ \Omega$$

Step III Calculation of $\mathbf{I}_L$

$$\text{I}_L = \frac{18.2\angle -15.95^\circ}{20.15\angle 29.74^\circ + 40} = 0.31\angle -25.81^\circ\ \text{A}$$



15) In the network shown in Fig., what load Z_L will receive the maximum power. Also find maximum power.



SOL:

Step I Calculation of $\mathbf{V}_m$

$$\mathbf{V}_m = \frac{\mathbf{V}_1\mathbf{Y}_1 + \mathbf{V}_2\mathbf{Y}_2}{\mathbf{Y}_1 + \mathbf{Y}_2} = \frac{(50\angle 0^\circ)\left(\frac{1}{5+j5}\right) + (25\angle 90^\circ)\left(\frac{1}{3-j4}\right)}{\frac{1}{5+j5} + \frac{1}{3-j4}} = 9.81\angle -78.69^\circ \text{V}$$

Step II Calculation of $\mathbf{Z}_m$

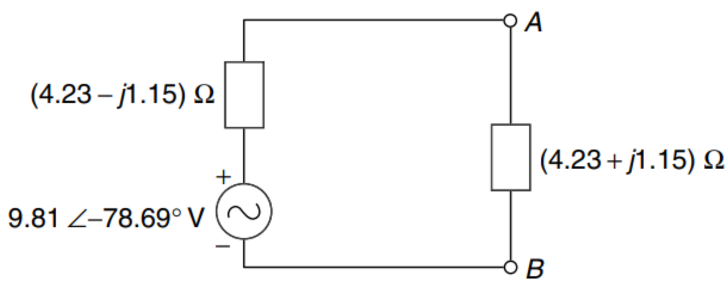
$$\mathbf{Z}_m = \frac{1}{\mathbf{Y}_m} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2} = \frac{1}{\frac{1}{5+j5} + \frac{1}{3-j4}} = 4.39\angle -15.26^\circ\ \Omega = (4.23 - j1.15)\ \Omega$$

Step III Calculation of $\mathbf{Z}_L$

For maximum power transfer $\quad \mathbf{Z}_L = \mathbf{Z}^*_{\text{Th}} = (4.23 + j1.15)\ \Omega$

Step IV Calculation of $P_{\max}$

$$P_{\max} = \frac{|\mathbf{V}_m|^2}{4R_L} = \frac{(9.81)^2}{4 \times 4.23} = 5.69 \text{ W}$$



6. TELLEGEN'S THEOREM

Statement:

Tellegen's theorem states that the algebraic sum of powers in all branches in a network at any instant is zero. This theorem is valid for any network that may be linear or non-linear, active or passive and time varying or time invariant, and all branch currents and voltages in the network must satisfy Kirchhoff's laws.

According to Tellegen's theorem, the rate of supply of energy by the active elements of a network equals the rate of energy dissipated or stored by the passive elements of the network. Tellegen's theorem is stated mathematically as follows:

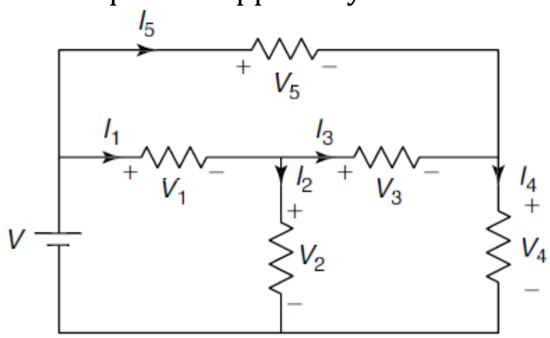
$$\sum_{k=1}^{b} V_k I_k = 0$$

V_k and I_k should satisfy KVL and KCL, respectively. In the above expression 'b' indicates the number of branches.

[Note: Total Power delivered by source = Total power absorbed by sinks]

PROBLEMS ON TELLEGEN'S THEOREM

16) Determine the power supplied by the source in the network



$V_1 = 5$ V, $I_1 = 2$ A

$V_2 = 10$ V, $I_2 = 0.8$ A

$V_3 = 3$ V, $I_3 = 6$ A

$V_4 = 7$ V, $I_4 = 3$ A

$V_5 = 8$ V, $I_5 = 4$ A

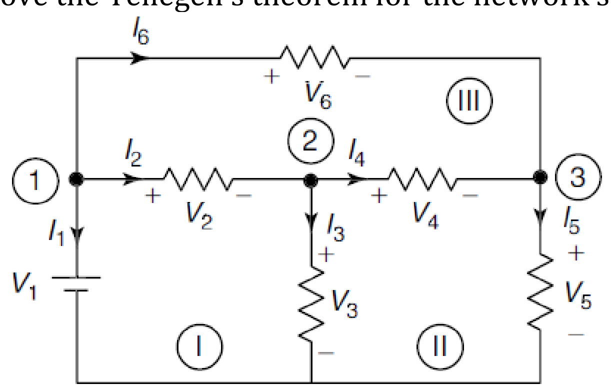
SOL:

By Tellegen's theorem,

$$\text{Power supplied by the source} = V_1 I_1 + V_2 I_2 + V_3 I_3 + V_4 I_4 + V_5 I_5$$

$$= 5 \times 2 + 10 \times 0.8 + 3 \times 6 + 7 \times 3 + 8 \times 4 = 89 \text{ W}$$

17) Prove the Tellegen's theorem for the network shown in the fig.



$V_1 = 8$ V, $V_2 = 4$ V, $V_4 = 2$ V

$I_1 = 4$ A, $I_2 = 2$ A, $I_3 = 1$ A

SOL:

Applying KVL to Mesh 1, $V_1 - V_2 - V_3 = 0$

$$V_3 = V_1 - V_2 = 8 - 4 = 4 \text{ V}$$

Applying KVL to Mesh 2, $V_3 - V_4 - V_5 = 0$

$$V_5 = V_3 - V_4 = 4 - 2 = 2 \text{ V}$$

Applying KVL to Mesh 3, $-V_6 + V_4 + V_2 = 0$

$$V_6 = V_2 + V_4 = 4 + 2 = 6 \text{ V}$$

Applying KCL at Node 1, $I_1 + I_2 + I_6 = 0$

$$I_6 = -I_1 - I_2 = -4 - 2 = -6 \text{ A}$$

Applying KCL at Node 2, $I_2 = I_3 + I_4$

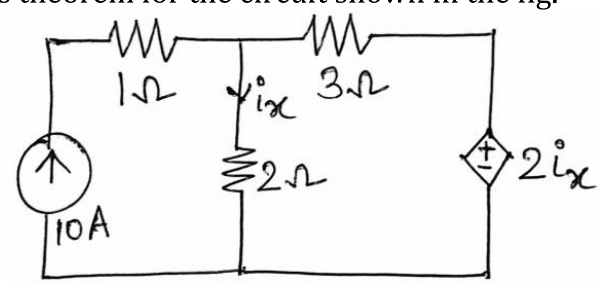
$$I_4 = I_2 - I_3 = 2 - 1 = 1 \text{ A}$$

Applying KCL at Node 3, $I_5 = I_4 + I_6 = 1 + (-6) = -5 \text{ A}$

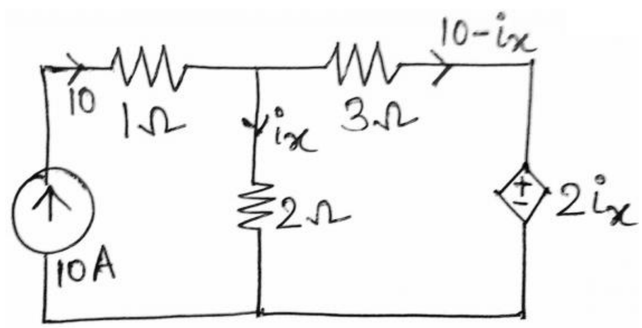
$$\sum_{b=1}^{6} V_b\, I_b = V_1 I_1 + V_2 I_2 + V_3 I_3 + V_4 I_4 + V_5 I_5 + V_6 I_6 = 8\times4 + 4\times2 + 4\times1 + 2\times1 + 2\times(-5) + 6\times(-6) = 0$$

Hence, Tellegen's theorem is verified.

18) Verify the Tellegen's theorem for the circuit shown in the fig.



SOL:

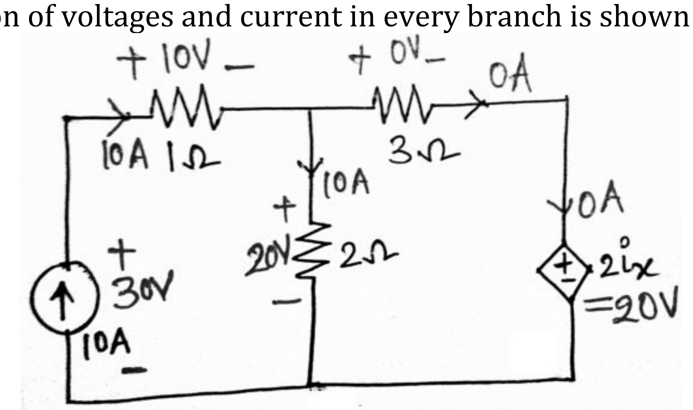


Apply KVL in the second loop

$$-2i_x + 3(10 - i_x) + 2i_x = 0$$

$$i_x = \frac{30}{3} = 10\text{A}$$

The distribution of voltages and current in every branch is shown in the fig.

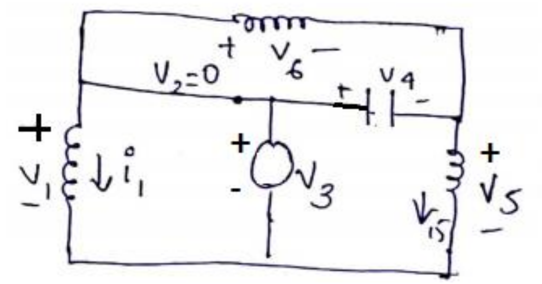


$$\sum P_{sources} = 10 \times 30 + 20 \times 0 = 300\,W$$

$$\sum P_{sinks} = 10 \times 10 + 8 \times 0 + 20 \times 10 = 300\,W$$

Hence Tellegen’s theorem is verified.

19) Verify the Tellegen’s theorem for the network shown in the fig.



$V_1 = 5\,V$, $V_4 = 2\,V$, $V_2 = 0$,

$i_1 = 2\,A$, $i_3 = 1\,A$.

SOL:

Applying KVL

$v_3 = v_1 = 5\,V.$

$v_5 = v_3 - v_4 = 3\,V.$

$v_6 = -(v_2 + v_4) = -2\,V$

$v_2 = 0 \quad i_2 = 0.$

Applying KCL

$i_6 = i_1 + i_2 = 2\,A$

$i_2 = i_3 + i_4 \quad i_4 = -i_3 = -1\,A$

$i_5 = i_4 - i_6 = -3\,A.$

Therefore $(5)\cdot(2) + (0)(0) + (5)(1) + (2)(-1) + (3)(-3) + (-2)\cdot(2)$

$= 10 + 0 + 5 - 2 - 9 - 4 = 0$

Hence Tellegen’s theorem is verified.

7. RECIPROCITY THEOREM

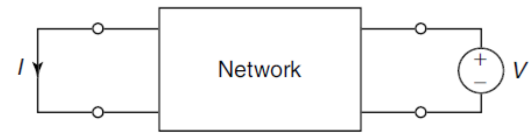
Statement:

It states that ‘in a linear, bilateral, active, single source network, the ratio of excitation to response remains same when the positions of excitation and response are interchanged.’

In other words, it may be stated as ‘if a single voltage source Va in the branch ‘a’ produces a current I_b in the branch ‘b’ then if the voltage source Va is removed and inserted in the branch ‘b’, it will produce a current I_b in branch ‘a’’.

Explanation Consider a network shown in Fig.

When the voltage source V is applied at the port 1, it produces a current I at the port 2. If the positions of the excitation (source) and response are interchanged, i.e., if the voltage source is applied at the port 2 then it produces a current I at the port 1.

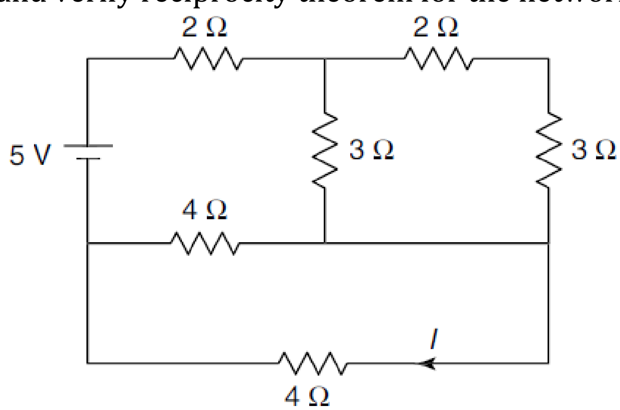


The limitation of this theorem is that it is applicable only to a single-source network. This theorem is not applicable in the network which has a dependent source. This is applicable only in linear and bilateral networks. In the reciprocity theorem, position of any passive element (R, L, C) do not change. Only the excitation and response are interchanged.

Steps to be followed in Reciprocity Theorem
1. Identify the branches between which reciprocity is to be established.
2. Find the current in the branch when excitation and response are not interchanged.
3. Find the current in the branch when excitation and response are interchanged.

PROBLEMS ON RECIPROCITY THEOREM

20) Find the current 'I' and verify reciprocity theorem for the network shown in Fig.



SOL:

Case I Calculation of the current I when excitation and response are not interchanged

Applying KVL to Mesh 1,

$$5 - 2I_1 - 3(I_1 - I_2) - 4(I_1 - I_3) = 0$$

$$9I_1 - 3I_2 - 4I_3 = 5 \quad \ldots\text{(i)}$$

Applying KVL to Mesh 2,

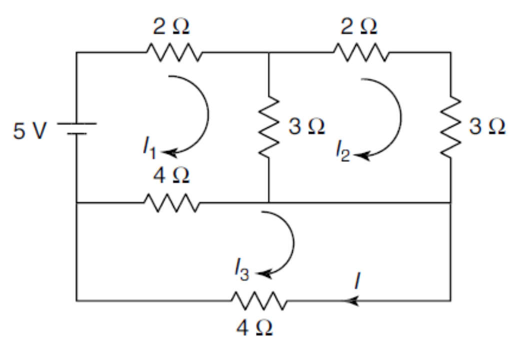
$$-3(I_2 - I_1) - 2I_2 - 3I_2 = 0$$

$$-3I_1 + 8I_2 = 0 \quad \ldots\text{(ii)}$$

Applying KVL to Mesh 3,

$$-4(I_3 - I_1) - 4I_3 = 0$$

$$-4I_1 + 8I_3 = 0 \quad \ldots\text{(iii)}$$



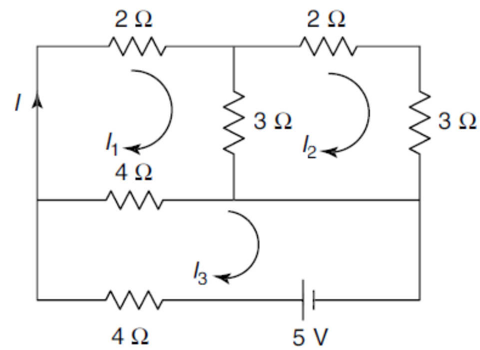
Solving Eqs (i), (ii) and (iii), $I_1 = 0.85$ A

$I_2 = 0.32$ A

$I_3 = 0.43$ A

$I = I_3 = 0.43$ A

Case II Calculation of current I when excitation and response are interchanged



Applying KVL to Mesh 1,

$$-2I_1 - 3(I_1 - I_2) - 4(I_1 - I_3) = 0$$

$$9I_1 - 3I_2 - 4I_3 = 0 \qquad \ldots(\text{i})$$

Applying KVL to Mesh 2,

$$-3(I_2 - I_1) - 2I_2 - 3I_2 = 0$$

$$-3I_1 + 8I_2 = 0 \qquad \ldots(\text{ii})$$

Applying KVL to Mesh 3, $-4(I_3 - I_1) + 5 - 4I_3 = 0$

$$-4I_1 + 8I_3 = 5 \qquad \ldots(\text{iii})$$

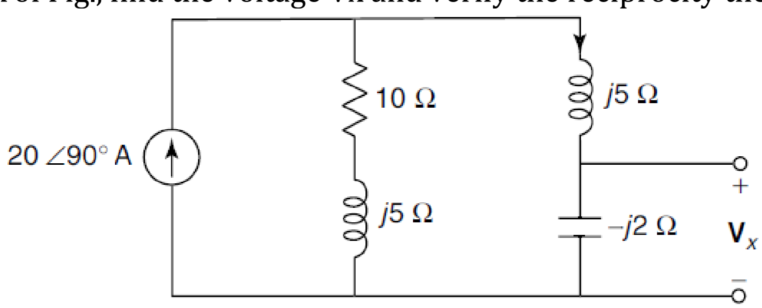
Solving Eqs (i), (ii) and (iii), $I_1 = 0.43$ A

$I_2 = 0.16$ A

$I_3 = 0.84$ A

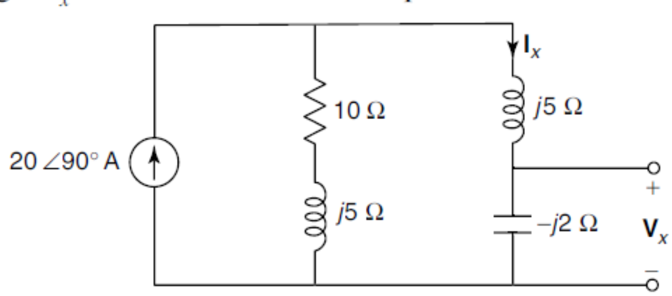
$I = I_1 = 0.43$ A

21) In the network of Fig., find the voltage Vx and verify the reciprocity theorem.



SOL:

Case I Calculation of voltage $\mathbf{V}_x$ when excitation and response are interchanged.

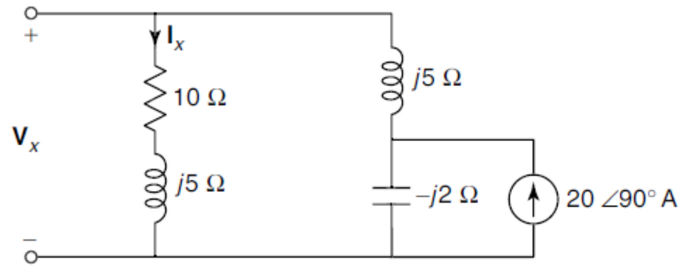


By current division rule,

$$\mathbf{I}_x = (20\angle 90^\circ)\frac{(10+j5)}{(10+j5)+(j5-j2)} = 17.46\angle 77.91^\circ \text{ A}$$

$$\mathbf{V}_x = (-j2)\mathbf{I}_x = (-j2)(17.46\angle 77.91^\circ) = 34.92\ \angle -12.09^\circ \text{ V}$$

Case II Calculation of voltage $\mathbf{V}_x$ when excitation and response are interchanged



$$\mathbf{I}_x = (20\angle 90^\circ)\frac{(-j2)}{(-j2)+(10+j5+j5)} = 3.12\angle -38.66^\circ \text{A}$$

$$\mathbf{V}_x = (10+j5)\mathbf{I}_x = (10+j5)(3.12\angle -38.66^\circ) = 34.88\angle -12.09^\circ \text{V}$$

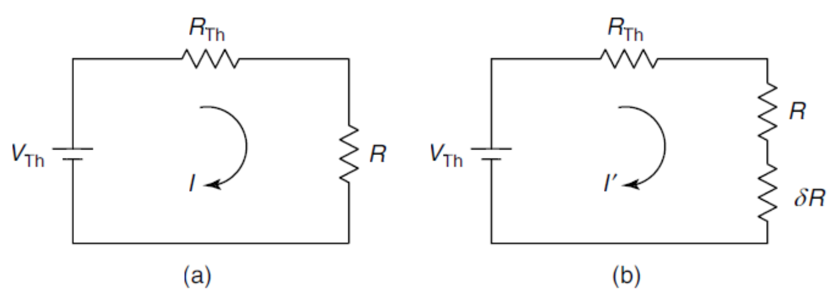
Since the voltage $\mathbf{V}_x$ is same in both the cases, the reciprocity theorem is verified.

8. COMPENSATION THEOREM

Statement:

It states that 'in any linear bilateral active network, if any branch carrying a current 'I' has its resistance R changed by an amount δR, the resulting changes that occur in the other branches are the same as those which would have been produced by an opposing voltage source of value Vc(IδR) introduced into the modified branch.'

Explanation Consider a network shown in Fig.(a), having load resistance R.



$$I = \frac{V_{Th}}{R_{Th} + R}$$

If the load resistance R be changed to $R + \delta R$ as shown in Fig.(b) then the current flowing in the circuit is

$$I' = \frac{V_{Th}}{R_{Th} + R + \delta R}$$

The change in the current is $\delta I = I' - I = \frac{V_{Th}}{R_{Th} + R + \delta R} - \frac{V_{Th}}{R_{Th} + R}$

$$= -\frac{V_{Th}\ \delta R}{(R_{Th} + R + \delta R)\,(R_{Th} + R)}$$

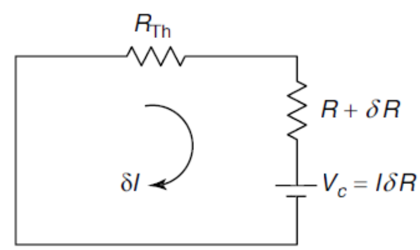
$$= -\frac{V_{Th}}{R_{Th} + R}\,\frac{\delta R}{R_{Th} + R + \delta R}$$

$$= -\frac{I\ \delta R}{R_{Th} + R + \delta R}$$

$$= -\frac{V_c}{R_{Th} + R + \delta R}$$

where $V_c = I\ \delta R$ and is called the *compensation voltage*.

δI has the same direction as I. This shows that the change in current δI due to a change in any branch in a linear network can be calculated by determining the current in that branch in a network obtained from the original network by removing all the independent sources and placing a voltage source called compensation source in series with the branch whose value is $V_c = I\ \delta R$, whose I is the current through the branch before its resistance is changed and δR is the change in resistance.

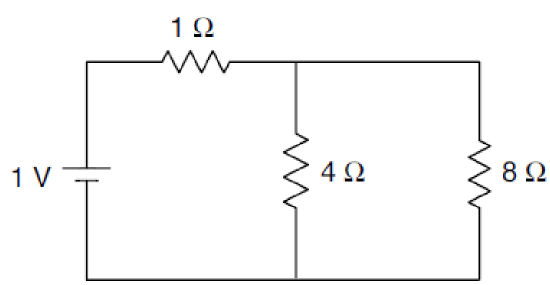


For AC circuit:

The compensation theorem states that 'In any linear bilateral active network, if any branch carrying a current 'I' has its impedance Z changed by an amount δZ, the resulting changes that occur in the other branches are the same as those which would have been provided by an opposing voltage source of value V_C(IδZ) introduced into the modified branch.'

PROBLEMS ON COMPENSATION THEOREM

22) In the network of Fig., the resistance of 4 Ω is changed to 2 Ω . Verify the compensation theorem.



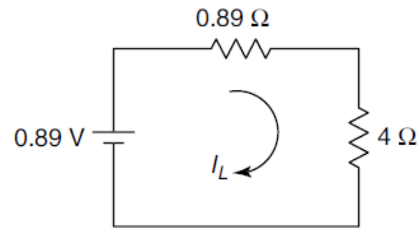
SOL:

Step I Calculation of change in current

Finding Thevenin's equivalent network across the 4 Ω resistor,

$$V_{Th} = 1 \times \frac{8}{8+1} = 0.89 \text{ V}$$

$$R_{Th} = \frac{1 \times 8}{1+8} = 0.89 \ \Omega$$



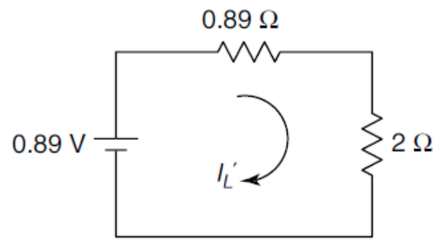
Thevenin's equivalent network is shown in Fig.

$$I_L = \frac{0.89}{0.89+4} = 0.18 \text{ A}$$

When resistance of 4 Ω is changed to 2 Ω,

$$I_L' = \frac{0.89}{0.89+2} = 0.31 \text{ A}$$

$$\delta I_L = I_L' - I_L = 0.31 - 0.18 = 0.13 \text{ A}$$



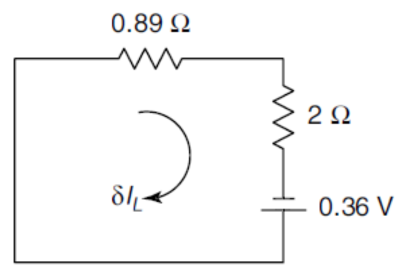
Step II Calculation of change of current by compensation theorem

$$\delta R = 2 - 4 = -2 \ \Omega$$

$$R + \delta R = 4 - 2 = 2 \ \Omega$$

$$V_c = I_L \delta R = 0.18 \times (-2) = -0.36 \text{ V}$$

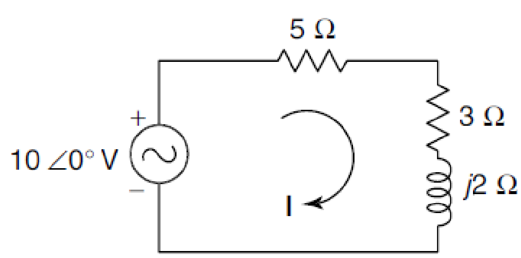
The compensating network is shown in Fig.



$$\delta I_L = \frac{0.36}{0.89+2} = 0.12 \text{ A}$$

Since change in current is same in both the steps, the compensation theorem is verified.

23) In the network of Fig., the impedance (3+j2) Ω is changed to (5+j3) Ω. Determine the change in current drawn from the source by direct calculation and then verify the result by compensation theorem.



SOL:

Step I Calculation of change in current

$$\mathbf{I} = \frac{10\angle 0^\circ}{5+3+j2} = 1.21\angle -14.04^\circ \text{ A}$$

When impedance $(3 + j2)\ \Omega$ is changed to $(5+j3)\ \Omega$,

$$\mathbf{I}' = \frac{10\angle 0^\circ}{5+5+j3} = 0.96\angle -16.7^\circ \text{ A}$$

$$\delta\mathbf{I} = \mathbf{I}' - \mathbf{I} = 0.96\angle -16.7^\circ - 1.21\angle -14.04^\circ = 0.25\angle 176.02^\circ \text{ A}$$

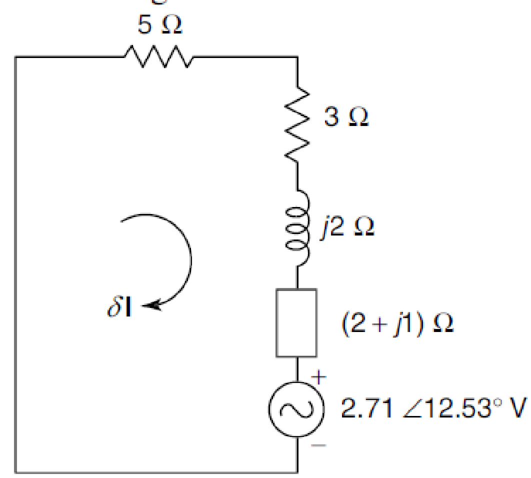
Step II Calculation of change in current by compensation theorem

$$\delta\mathbf{Z} = (5+j3) - (3+j2) = (2+j1)\ \Omega$$

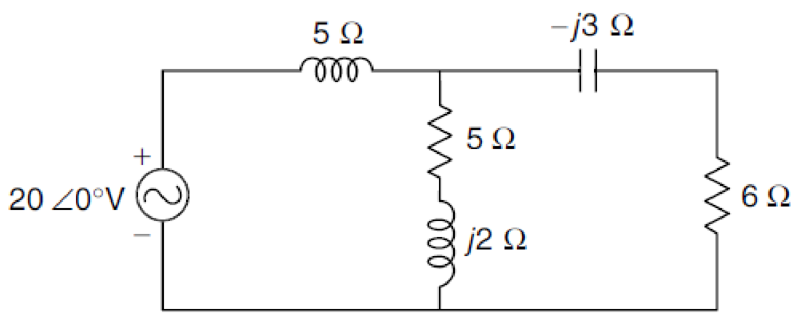
$$\mathbf{V}_c = \delta\mathbf{Z}\mathbf{I} = (2+j1)(1.21\angle -14.04^\circ) = 2.71\angle 12.53^\circ \text{ V}$$

The compensating network is as shown in Fig.

$$\delta\mathbf{I} = -\frac{2.71\angle 12.53^\circ}{5+3+j2+2+j1} = 0.26\angle 175.83^\circ \text{ A}$$



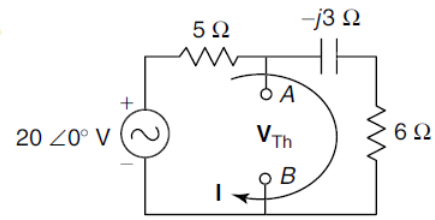
24) In the network of Fig., the impedance (5 + j2) Ω is changed to (1 + j1) Ω. Find the change in current drawn from the supply by direct calculation and then verify by the compensation theorem.



SOL:

Step I Calculation of change in current

Finding Thevenin's equivalent circuit across impedanc of (5 + *j*2) Ω

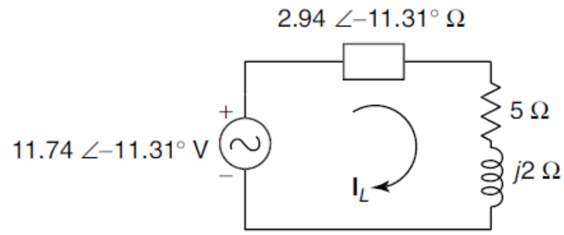


$$\mathbf{Z}_{\text{Th}} = \frac{5(6-j3)}{5+6-j3} = 2.94\angle -11.31°\ \Omega$$

$$\mathbf{I} = \frac{20\angle 0°}{5+6-j3} = 1.75\angle 15.26°\ \text{A}$$

$$\mathbf{V}_{\text{Th}} = (6-j3)\mathbf{I} = (6-j3)(1.75\angle 15.26°) = 11.74\angle -11.31°\ \text{V}$$

Thevenin's equivalent network is shown in Fig.



$$\mathbf{I}_L = \frac{11.74\angle -11.31°}{2.94\angle -11.31° + 5 + j2} = 1.47\angle -21.55°\ \text{A}$$

When impedance (5 + *j*2) Ω is changed to (1 + *j*1) Ω,

$$\mathbf{I}'_L = \frac{11.74\angle -11.31°}{2.94\angle -11.31° + 1 + j1} = 3\angle -17.53°\ \text{A}$$

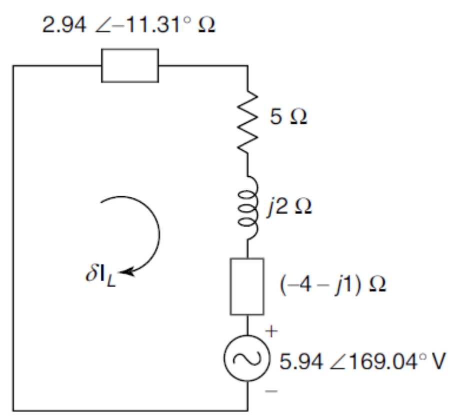
$$\delta \mathbf{I}_L = \mathbf{I}'_L - \mathbf{I}_L = 3\angle -17.53° - 1.44\angle -25° = 1.53\angle -13.69°\ \text{A}$$

Step II Calculation of change in current by compensation theorem

$$\delta \mathbf{Z} = (1+j1) - (5+j2) = (-4-j1)\ \Omega$$

$$\mathbf{V}_c = \mathbf{I}_L \delta \mathbf{Z} = (1.44\angle -25°)(-4-j1) = 5.94\angle 169.04°\ \text{V}$$

The compensating network is shown in Fig.



$$\delta \mathbf{I}_L = -\frac{5.94\angle 169.04°}{2.94\angle -11.31° + 5 + j2 - 4 - j1}$$

$$= 1.52\angle -17.18°\ \text{A}$$